III.1 Amalgamated product

정의 1 G, G_1, G_2 : groups with $f_i : G \to G_i$, homomorphism for $i = 1, 2$.
$F :=$ the free group generated by $G_1 \amalg G_2$. Denote by $x \cdot y$ the product in F.
$R :=$ the normal subgroup of F generated by the words $(xy) \cdot y^{-1} \cdot x^{-1}$,
where both $x, y \in G_i$, $i = 1, 2$ and $f_1(z) \cdot f_2(z)^{-1}$ for $z \in G$.
The amalgamated product of G_1 and G_2 over G, $G_1 \underset{G}{*} G_2 := F/R$

Remark. $G_1 * G_2 := G_1 \underset{\{1\}}{*} G_2$

Note.

$$\begin{tikzcd}
 & G_1 \arrow[dr, "g_1"] & \\
G \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \circlearrowright & G_1 \underset{G}{*} G_2 \\
 & G_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

where the canonical map $g_i : G_i \overset{i}{\hookrightarrow} F \xrightarrow{p} F/R = G_1 \underset{G}{*} G_2$ is a homomorphism.

Then $g_1 f_1(z) = g_2 f_2(z)$ by the definition of F/R.

2. universal property of amalgamation

(a) Suppose $h_i : G_i \to H$ is a homomorphism with $h_1 f_1 = h_2 f_2$.
Then $\exists!$ homomorphism $h : G_1 \underset{G}{*} G_2 \to H$ such that $h g_i = h_i$, $i = 1, 2$.

(b) If H is generated by $h_1(G_1) \cup h_2(G_2)$, then h is onto.

$$\begin{tikzcd}
 & G_1 \arrow[dr, "g_1"] \arrow[drr, "h_1"] & & \\
G \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \circlearrowright & G_1 \underset{G}{*} G_2 \arrow[r, "h"] & H \\
 & G_2 \arrow[ur, "g_2"'] \arrow[urr, "h_2"'] & &
\end{tikzcd}$$

증명
(a) Define $h' : F \to H$ as the unique homomorphism determined by the condition $h'|_{G_i} = h_i$.

$R \subset \ker(h')$:

R의 generator $(xy) \cdot y^{-1} \cdot x^{-1}$,$f_1(z) \cdot f_2(z)^{-1}$에 대해서만 check하면 된다.
$h_1(xy)h_1(y^{-1})h_1(x^{-1}) = 1$ for $x, y \in G_1$,
$h_2(xy)h_2(y^{-1})h_2(x^{-1}) = 1$ for $x, y \in G_2$ and
$h_1(f_1(z))h_2(f_2(z)^{-1}) = h_1 f_1(z)(h_2 f_2(z))^{-1} = 1$ $(\because h_1 f_1 = h_2 f_2)$

$\therefore \exists h : F/R \to H$.

G_i 상에서 $h' = h_i \Rightarrow h_i = h'i = hpi = hg_i \Rightarrow hg_i = h_i$

$G_i \;\overset{i}{\hookrightarrow} F \overset{h'}{\to}\; H$
$g_i \searrow\curvearrowright\downarrow p \curvearrowright\nearrow h$: diagram commutes from definitions of g_i and h.
$G_1 \underset{G}{*} G_2$
F/R

Uniqueness is obvious since h is already determined on $g_i(G_i)$
and $g_1(G_1) \cup g_2(G_2)$ generates $G_1 \underset{G}{*} G_2$.

(b) Suppose $\forall a \in H$ is $a = a_1 a_2 \cdots a_k$.
$a_j \in h_1(G_1) \cup h_2(G_2) \Rightarrow a_j = h_i(x_j)$ for $i = i(a_j) = 1$ or 2
$\Rightarrow a = \Pi a_j = \Pi h_i(x_j) = \Pi h g_i(x_j) = h(\Pi g_i(x_j))$ □

숙제 9.
(1) $G_1 \underset{G}{*} G_2 \cong G_1 * G_2/\Gamma$,
where Γ is the normal subgroup generated by $f_1(z) \cdot f_2(z)^{-1}$.
(2) f_1 : onto $\Rightarrow$ g_2 : onto
(3) f_1 : $\cong$ $\Rightarrow$ g_2 : $\cong$

3. Group presentation.

정리 1 $G_1 =< x_1, \cdots , x_n | r_1, \cdots , r_k >$

$G_2 =< y_1, \cdots , y_m | s_1, \cdots , s_l >$

$$G =< z_1, \cdots , z_p | \ any > \ with \begin{tikzcd} & G_1 \\ G \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & G_2 \end{tikzcd}$$

$\Rightarrow G_1 \underset{G}{*} G_2 \cong < x_1, \cdots , x_n, y_1, \cdots , y_m | r_1, \cdots , r_k, s_1, \cdots , s_l \ and \ f_1(z_i)f_2(z_i)^{-1}$
$\quad for \ i = 1, \cdots , p >$

증명 Let $\overline{G} =< x_1, \cdots , x_n, y_1, \cdots , y_m | r_1, \cdots , r_k, s_1, \cdots , s_l$ and $f_1(z_i)f_2(z_i)^{-1}$
for $i = 1, \cdots , p >$.

Define $\phi : \overline{G} \to G_1 \underset{G}{*} G_2$ by

$$x_i \mapsto g_1("x_i") \quad ("x_i" \in G_1)$$
$$y_i \mapsto g_2("y_i") \quad ("y_i" \in G_2)$$

$$r_i = x_{i_1} \cdots x_{i_k} \text{ in } \overline{G} \Rightarrow \phi(r_i) = \phi(x_{i_1} \cdots x_{i_k}) = g_1("x_{i_1}") \cdots g_1("x_{i_k}")$$
$$= g_1("x_{i_1} \cdots x_{i_k}") = g_1("r_i")$$
$$= g_1(1) = 1$$

Similarly $\phi(s_j) = 1$

$$\phi(f_1(z_i)f_2(z_i)^{-1}) = g_1(f_1(z_i))g_2(f_2(z_i))^{-1} = 1$$

$\Rightarrow \phi$ is a homomorphism.

For the converse, use universal property:

아래 diagram에서 $h_1(x_i) = x_i, h_2(y_j) = y_j$라 두면 $\overline{G}$의 정의에 의해 $h_1 f_1(z_i) = h_2 f_2(z_i)$가 성립하고 따라서 universal property에 의해

$$\exists! h : G_1 \underset{G}{*} G_2 \to \overline{G} \text{ s.t. } \begin{tikzcd} & G_1 \arrow[dr, "g_1"] \arrow[drr, "h_1"] & & \\ G \arrow[ur, "f_1"] \arrow[dr, "f_2"'] \arrow[r, phantom, "\curvearrowright"] & & G_1 \underset{G}{*} G_2 \arrow[r, "h"] & \overline{G} \\ & G_2 \arrow[ur, "g_2"'] \arrow[urr, "h_2"'] & & \end{tikzcd} \quad \text{commute.}$$

Now $h\phi(x_i) = hg_1(x_i) = h_1(x_i) = x_i$ and
$h\phi(y_j) = hg_2(y_j) = h_2(y_j) = y_j$
$\Rightarrow \quad h\phi = $id.

$\phi h(g_1(x_i)) = \phi h_1(x_i) = \phi(x_i) = g_1(x_i)$ and
$\phi h(g_2(y_j)) = \phi h_2(y_j) = \phi(y_j) = g_2(y_j)$
$\Rightarrow \quad \phi h = \mathrm{id}$.

$\therefore \phi$ is an isomorphism with inverse h. $\square$

4. Special Case

$G_1 = \{1\}$ (or $G_2 = \{1\}$) $\Rightarrow G_1 \underset{G}{*} G_2 = G_2 / < f_2(G) >$[1]

[1] $< f_2(G) >$ is the normal subgroup generated by $f_2(G)$