III.2 Van Kampen theorem

정리 1 ***(Van-Kampen theorem)***
Suppose $X = U \cup V$ with $x_0 \in U \cap V$ where U, V and $U \cap V$ are open and path-connected . Then $\pi_1(X, x_0) \cong \pi_1(U, x_0) \underset{\pi_1(U\cap V, x_0)}{*} \pi_1(V, x_0)$ where the homomorphism for an amalgamated product are induced by inclusions.

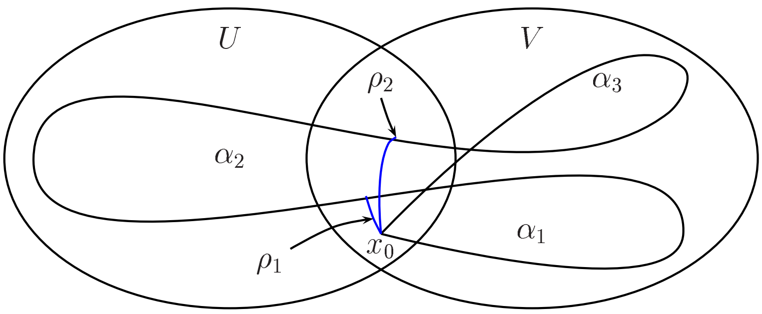
$$
\begin{tikzcd}
 & U \arrow[dr, "j_1"] & \\
U \cap V \arrow[ur, "i_1"] \arrow[dr, "i_2"'] \arrow[rr, phantom, "\curvearrowright"] & & X \\
 & V \arrow[ur, "j_2"'] &
\end{tikzcd}
\quad \overset{\pi_1}{\rightsquigarrow} \quad
\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "j_{1\sharp}"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_{1\sharp}"] \arrow[dr, "i_{2\sharp}"'] \arrow[rr, phantom, "\curvearrowright"] & & \pi_1(X, x_0) \\
 & \pi_1(V, x_0) \arrow[ur, "j_{2\sharp}"'] &
\end{tikzcd}
$$

증명
$\pi_1(U, x_0) \underset{\pi_1(U\cap V, x_0)}{*} \pi_1(V, x_0) \cong \pi_1(X, x_0)$임을 보여야 한다.

$\exists! h : \pi_1(U, x_0) \underset{\pi_1(U\cap V, x_0)}{*} \pi_1(V, x_0) \to \pi_1(X, x_0)$, a homomorphism by universal prop.
Show h is (i) onto and (ii) one to one:

(i) h is onto : use 2(b), i.e., show $h_1\pi_1(U)$ and $h_2\pi_1(V)$ generate $\pi_1(X)$:

$\forall \alpha \in \pi_1(X)$, use Lebesgue lemma to obtain $\alpha = \alpha_1 * \cdots * \alpha_p$,
s.t. $\alpha_i := \alpha|_{[t_{i-1}, t_i]} \subset U$ or V.
Choose a path ρ_i from x_0 to $\alpha_i(t_i)$ lying completely in U or V
and let $\rho_p = \rho_0 =$ constant path x_0.
Let $\widetilde{\alpha_i} = \rho_{i-1} * \alpha_i * \overline{\rho_i}$.
Then $[\alpha] = [\alpha_1 * \cdots * \alpha_p] = [\widetilde{\alpha_1} * \cdots * \widetilde{\alpha_p}] = [\widetilde{\alpha_1}] \cdots [\widetilde{\alpha_p}]$ and
each $[\widetilde{\alpha_i}] \in h_1\pi_1(U)$ or $h_2\pi_1(V)$.

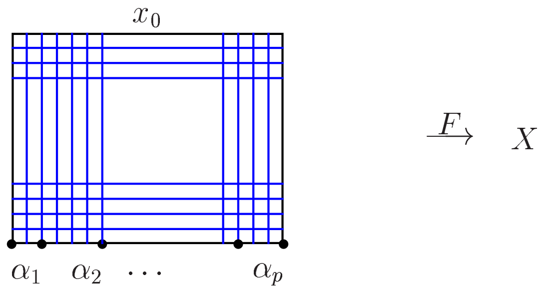


(ii) h is one-to-one i.e. show ker h =0:

Suppose $h([\alpha_1]\cdots[\alpha_p]) = 1$, where $[\alpha_i]$ in either $\pi_1(U)$ or $\pi_1(V)$ and show $[\alpha_1]\cdots[\alpha_p] = 0$ in $\pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$.

$\alpha_1 * \cdots * \alpha_p \sim 1$ in X by hypothesis

$\Rightarrow \exists F : I \times I \to X = U \cup V$ with $F_0 = \alpha_1 * \cdots * \alpha_p$ and $F_1 = x_0$.



Using Lebesgue lemma, can partition $I \times I$ fine enough so that

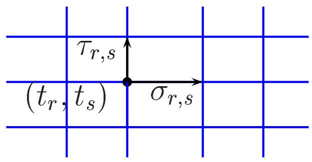
(1) $F(\text{each } \square) \subset U$ or V and

(2) partition contains end points of domain of each α_i so that

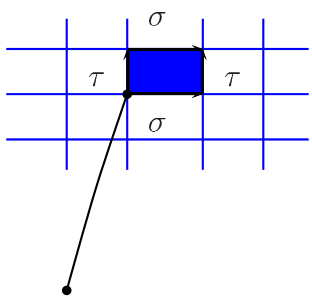
$$\alpha_1 = \alpha_{11} * \alpha_{12} * \cdots * \alpha_{1i_1},$$
$$\vdots$$
$$\alpha_p = \alpha_{p1} * \quad \cdots \quad * \alpha_{pi_p}.$$

Choose and fix a path $\rho_{r,s}$ from x_o to $F(t_r, t_s)$ once and for all s.t. $\rho_{r,s} \subset \{x_0\}, U \cap V, U, V$ respectively if $F(t_r, t_s) \in \{x_0\}, U \cap V, U, V$ respectively.

Let $\widetilde{\sigma_{r,s}} = \rho_{r,s} * \sigma_{r,s}, * \overline{\rho_{r+1,s}}$ and $\widetilde{\tau_{r,s}} = \rho_{r,s} * \tau_{r,s} * \overline{\rho_{r,s+1}}$,
where $\sigma_{r,s}$ and $\tau_{r,s}$ are short paths given by the following picture.



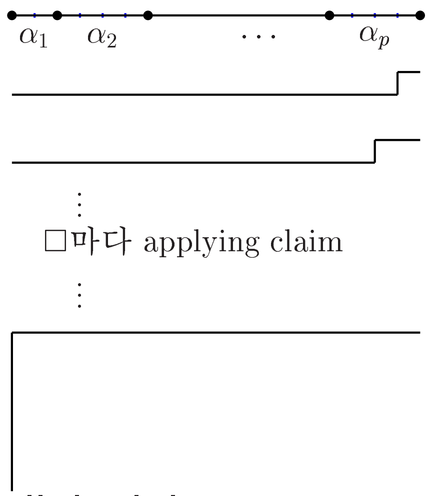
Claim: $\widetilde{\sigma_{r,s}} * \widetilde{\tau_{r+1,s}} \sim \widetilde{\tau_{r,s}} * \widetilde{\sigma_{r,s+1}}$ in U or V.
(Proof is clear as in the following picture.)

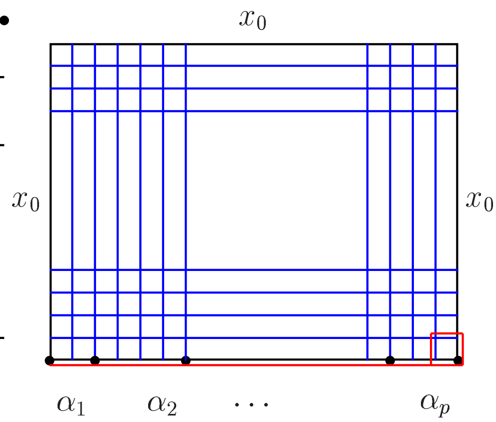


In $\pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$,

$[\alpha_1]\cdots[\alpha_p] = [\widetilde{\alpha}_{11}][\widetilde{\alpha}_{12}]\cdots[\widetilde{\alpha}_{1i_1}]\cdots[\widetilde{\alpha}_{p1}] \quad \cdots \quad [\widetilde{\alpha}_{pi_p}]$

$=$
$=$
$=$
$=$





$= [x_0][x_0]\cdots[x_0]$
$= [x_0] = 1$

$\square$