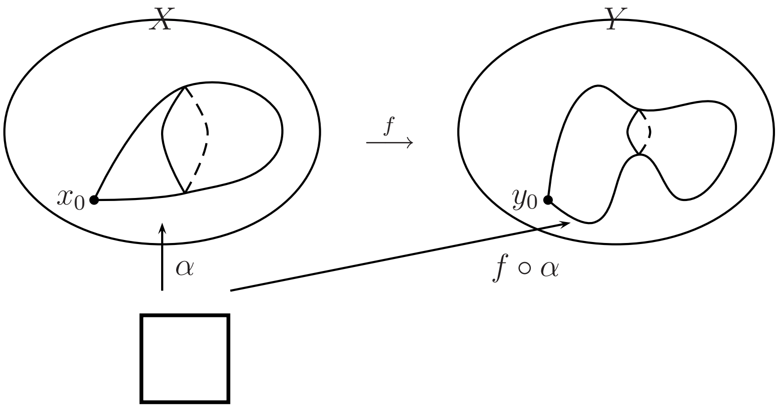
IV.2 Functorial property and Homotopy invariance

(1) Functorial property of π_n

$$\boxed{\begin{array}{ll} f:(X,x_0)\to(Y,y_0) \Rightarrow f_* = \pi_n(f): & \pi_n(X,x_0) \to \pi_n(Y,y_0) \\ & \quad [\alpha] \quad \mapsto \quad [f\circ\alpha] \end{array}}$$



1. f_* is a homomorphism : $f\circ(\alpha*\beta) = (f\circ\alpha)*(f\circ\beta)$

2. $\mathrm{id}_* = \mathrm{id}$: clear

$$(g\circ f)_* = g_*\circ f_* : \quad (X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (Z,z_0)$$
$$\Rightarrow \pi_n(X,x_0) \xrightarrow{f_*} \pi_n(Y,y_0) \xrightarrow{g_*} \pi_n(Z,z_0)$$
$$[\alpha] \quad \mapsto [f\circ\alpha] \mapsto [g\circ f\circ\alpha]$$

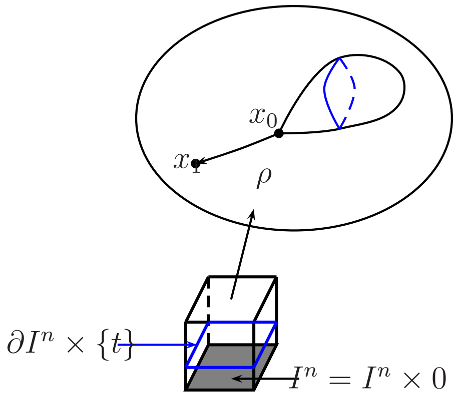
3. $f,g:(X,x_0)\to(Y,y_0)$ and $f \underset{H}{\simeq} g(\mathrm{rel}\ x_0)$

$\Rightarrow f_* = g_*$: The proof is the same as in π_1 -case, i.e.,

$$f\circ\alpha = H_0\circ\alpha \simeq H_1\circ\alpha = g\circ\alpha$$

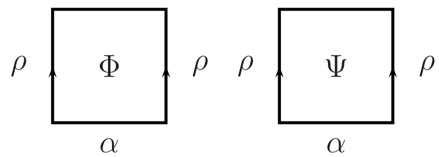
Change of Base point

Let $x_0, x_1 \in X$ and ρ be a path from x_0 to x_1.
Given $\alpha : (I^n, \partial I^n) \to (X, x_0)$, define $\Phi : I^n \times I \to X$ as an extension of a map $\phi : J = I^n \times 0 \cup \partial I^n \times I \to X$ defined by $\phi|_{I^n \times 0} = \alpha$ and $\phi|_{\partial I^n \times \{t\}} = \rho(t)$.
(Note that J is a strong deformation retract of I^{n+1} and hence any map defined on J has an extension on I^{n+1}.)

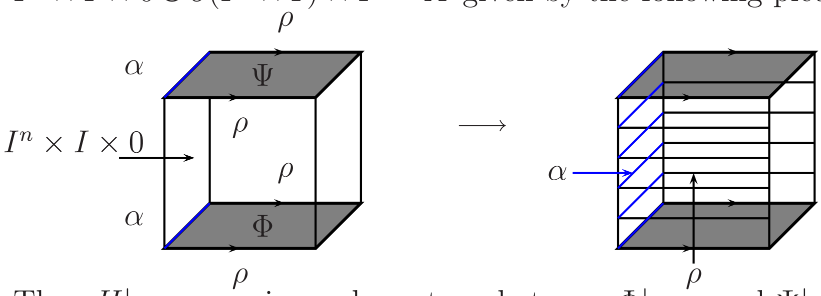


$$\begin{aligned} \text{Define } \phi_\rho : \pi_n(X, x_0) &\to \pi_n(X, x_1) \\ [\alpha] &\mapsto [\Phi|_{I^n \times \{1\}}] \end{aligned}$$

1. independent of choice of Φ:

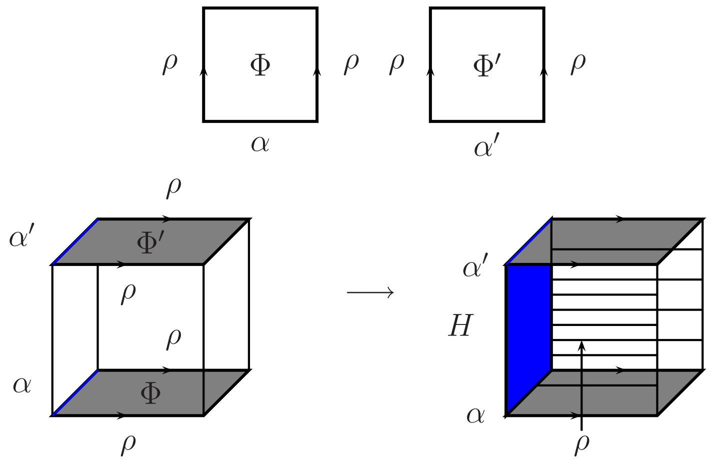
Let  be two extensions.

Define a homotopy H between Φ and Ψ as an extension of a map : $I^n \times I \times 0 \cup \partial(I^n \times I) \times I \to X$ given by the following pictures.

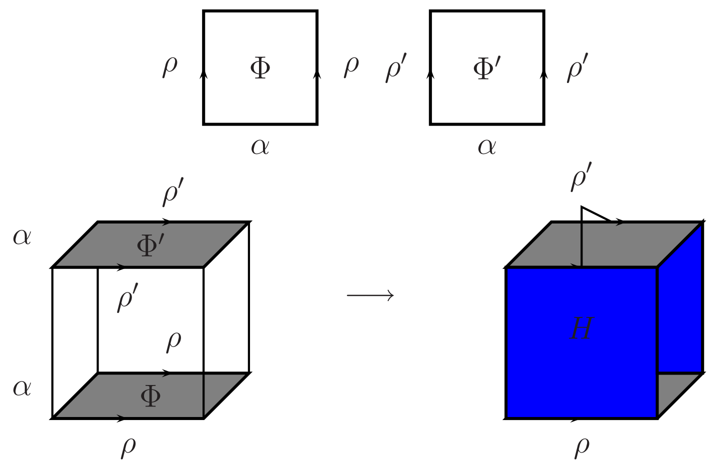


Then $H|_{I^n \times I \times \{1\}}$ gives a homotopy between $\Phi|_{I^n \times 1}$ and $\Psi|_{I^n \times 1}$.
$\therefore \Phi|_{I^n \times 1} \sim \Psi|_{I^n \times 1}$

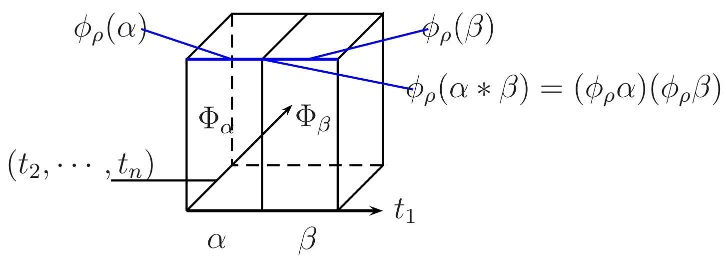
2. ϕ_ρ is well-defined i.e., $\alpha \underset{H}{\sim} \alpha' \Rightarrow \Phi|_{I^n\times 1} \sim \Phi'|_{I^n\times 1}$



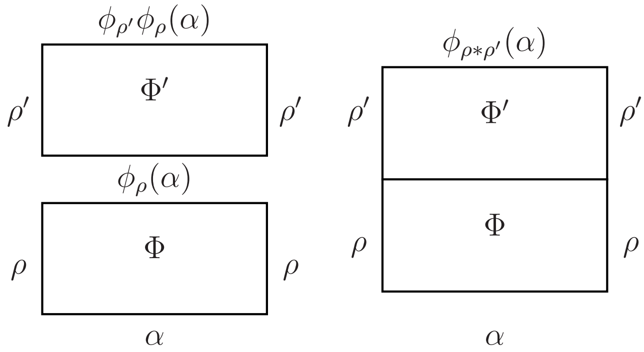
3. ϕ_ρ depends only on the homopoty class of $\rho(\text{rel } \partial)$ i.e., $\rho \underset{H}{\sim} \rho' \Rightarrow \phi_\rho = \phi_{\rho'}$, so that we can write as $\phi_{[\rho]}$.



4. $\phi_{[\rho]}$ is a homomorphism i.e., $\phi_{[\rho]}[\alpha * \beta] = \phi_{[\rho]}[\alpha]\phi_{[\rho]}[\beta]$.



5. $\phi_{\rho*\rho'} = \phi_{\rho'} \cdot \phi_\rho$ where $\rho(1) = \rho'(0)$:



6. ϕ_ρ is an isomorphism and $\phi_{\overline{\rho}} = \phi_\rho^{-1}$:
$\phi_\rho \cdot \phi_{\overline{\rho}} \overset{5}{=} \phi_{\overline{\rho}*\rho} \overset{3}{=} \phi_{x_1} = \mathrm{id}_{\pi_n(X,x_1)}$ and similarly $\phi_{\overline{\rho}} \cdot \phi_\rho = \mathrm{id}_{\pi_n(X,x_0)}$

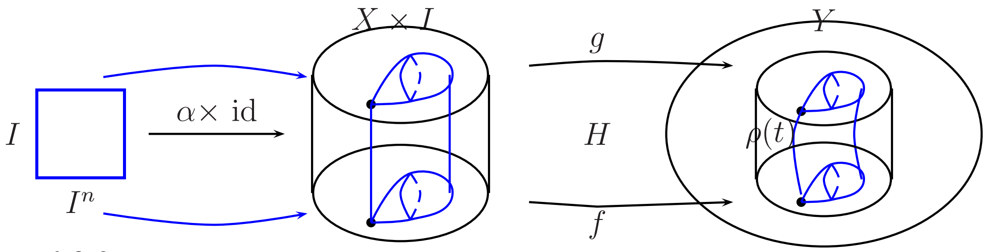
Remark. If ρ is a loop, then $\phi_\rho : \pi_n(X, x_0) \to \pi_n(X, x_0)$. Hence, we have a right action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

숙제 12. (1) X is n-simple.
$\overset{def}{\Leftrightarrow} \exists x_0 \in X$ s.t. $\pi_1(X, x_0)$ action on $\pi_n(X, x_0)$ is trivial.
$\Leftrightarrow \forall x \in X, \pi_1(X, x)$ action on $\pi_n(X, x)$ is trivial.
(2) X is 1-simple. $\Leftrightarrow \pi_1(X)$ is abelian.

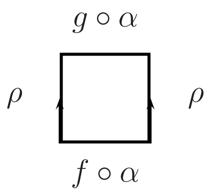
Homotopy invariance

1. $f \overset{H}{\simeq} g : X \to Y \quad \Rightarrow \quad$
$$\begin{tikzcd} \pi_n(X,x) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & \arrow[d, phantom, "\curvearrowright"] & \pi_n(Y,g(x)) \\ & \pi_n(Y,f(x)) \arrow[ur, "\phi_\rho :\cong"'] & \end{tikzcd}$$
where $\rho(t) = H(x,t)$

증명



$\Rightarrow$

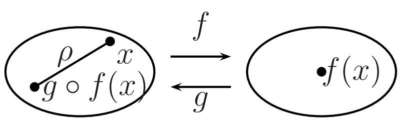


$\Rightarrow$
$$\begin{array}{ccc} [g\circ\alpha] & = & \phi_\rho[f\circ\alpha] \\ \| & & \| \\ g_*[\alpha] & & \phi_\rho \circ f_*[\alpha] \end{array}$$

$\square$

2. $f : X \to Y$ is a homotopy equivalence.
$\Rightarrow f_* : \pi_n(X,x) \to \pi_n(Y,y)$ is an isomorphism.

증명 exactly same as π_1-case:



$g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$
$g_* \circ f_* = (g \circ f)_* = \phi_\rho \circ id_* :\cong \ \Rightarrow g_*$ is onto.
마찬가지로 $f_* \circ g_* :\cong \ \Rightarrow g_*$ is 1-1.
$\therefore f_*$ is $\cong$.
(두번째 줄의 f_*와 세번째 줄의 f_*는 다르다.)

$\square$