VI.2 Properties of Simplicial Homology

정의 1 $f : |K| \to |L|$, a simplicial map.
Define $f_{\sharp,p} : C_p(K) \to C_p(L)$ by $f_{\sharp,p}[v_0, \cdots, v_p] = [f(v_0), \cdots, f(v_p)]$.
$f_\sharp$ is well-defined, i.e., $f(\overline{\sigma}) = -f(\sigma)$.
$\{f_{\sharp,p}\}$ is a "chain map" induced by f, i.e., $f_\sharp \circ \partial = \partial \circ f_\sharp$.

$$\begin{array}{ccccccccc}
\cdots & \to C_{p+1} & \xrightarrow{\partial} & C_p & \xrightarrow{\partial} & C_{p-1} & \xrightarrow{\partial} & \cdots \\
& \downarrow f_{p+1} & \curvearrowright & \downarrow f_p & \curvearrowright & \downarrow f_{p-1} & & \\
\cdots & \to D_{p+1} & \xrightarrow{\partial} & D_p & \xrightarrow{\partial} & D_{p-1} & \xrightarrow{\partial} & \cdots
\end{array}$$

$\{f_p\}$ is called a chain map if $\partial_p \circ f_p = f_{p-1} \circ \partial_p$

1. $f_\sharp$ commutes with ∂ and $f_\sharp$ induces a homomorphism $f_* : H_p(K) \to H_p(L)$.

증명
$$\begin{aligned}\partial \circ f_\sharp[v_0, \cdots, v_p] &= \partial[f(v_0), \cdots, f(v_p)] \\ &= \Sigma(-1)^i[f(v_0), \cdots, f(\hat{v_i}), \cdots, f(v_p)] \\ &= f_\sharp \partial[v_0, \cdots, v_p]\end{aligned}$$
$\partial \circ f_\sharp = f_\sharp \circ \partial \Rightarrow f_\sharp : Z_p \to Z_p$ and $B_p \to B_p \Rightarrow f_* : H_p \to H_p$. □

2. (i) $f : K \to L$ and $g : L \to M$: simplicial maps $\Rightarrow (g \circ f)_* = g_* \circ f_*$.
(ii) id : $K \to K \Rightarrow \mathrm{id}_* = \mathrm{id}$.

증명 clear from definition since $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$ and $\mathrm{id}_\sharp = \mathrm{id}$. □

Topological invariance of Simplicial Homology(key idea)

1. Let K' be a subdivision of K and let $\lambda : C_p(K) \to C_p(K')$ be an obvious subdivision operator. Then it can be shown that $\lambda_* : H_p(K) \to H_p(K')$ is an isomorphism.

2. K and L are simplicial complex structure for X. Choose a common subdivision M for K and L. Then from 1, $H_p(K) \cong H_p(M) \cong H_p(L)$.

Ordered Simplicial Homology

Ordered chain complex :
$\triangle_p(K)$:= the free abelian group generated by the ordered p-simplices in K and let $\partial(v_0, \cdots, v_p) = \Sigma(-1)^i(v_0, \cdots, \hat{v_i}, \cdots, v_p)$.

Then $\partial^2 = 0$: same as before.
$\Rightarrow$We have a chain complex $\{\triangle_p(K), \partial\}$ called the ordered chain complex.
$\cdots \xrightarrow{\partial} \triangle_{p+1}(K) \xrightarrow{\partial} \triangle_p(K) \xrightarrow{\partial} \triangle_{p-1}(K) \xrightarrow{\partial} \cdots$

$\Rightarrow H_p^{\triangle}(K) = \ker \partial_p / \operatorname{im}\partial_{p+1}$: ordered simplicial homology.