VII Singular Homology

VII.1 Categories and Functors

정의 1 A category $\mathcal{C}$ consists of
(1) A class of objects X
(2) $\forall$ ordered pair X, Y of objects, a set $hom(X,Y)$ of morphisms
(denoted by $f : X \to Y$) s.t. $\forall f \in hom(X,Y), g \in hom(Y,Z)$,
their composite $g \circ f \in hom(X,Z)$ is defined and satisfies
(associativity) $f \in hom(X,Y), g \in hom(Y,Z), h \in hom(X,Z)$
$\Rightarrow h \circ (g \circ f) = (h \circ g) \circ f$
($\exists$ of id)$\forall X$: object, $\exists 1_X \in hom(X,X)$ called an identity morphism
s.t. $1_X \circ f = f$ and $g \circ 1_X = g$,
$\forall f \in hom(Y,X)$ and $\forall g \in hom(X,Y)$ for $\forall Y$: object.

1. id morphism is unique. ($\because 1_X = 1_X \circ 1'_X = 1'_X$)

정의 2 $g \circ f = 1_X \Rightarrow g$ is called a left inverse of f.
f is called a right inverse of g.

2. If f has a left inverse g and right inverse g', then $g = g'$.
($\because g' = 1_X \circ g' = (g \circ f) \circ g' = g \circ (f \circ g') = g \circ 1_X = g$)
f has an inverse. $\Rightarrow f$ is called an equivalence.

정의 3 A covariant(contravariant resp.) functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a function assigning to each object X of $\mathcal{C}$, an object $F(X)$ of $\mathcal{D}$ and assigning to each morphism $f : X \to Y$, a morphism
$F(f) : F(X) \to F(Y)$ s.t. (1) $F(1_X) = 1_{F(X)}, \forall X$
($\leftarrow$ resp.) (2) $F(g \circ f) = F(g) \circ F(f) (= F(f) \circ F(g)$ resp.)

Note. f : equivalence. $\Rightarrow F(f)$: equivalence.
($\because F(g) \circ F(f) = F(g \circ f) = F(1_X) = 1_{F(X)}$)

Example The category of sets and functions (=$\mathcal{S}$)
The category of topological spaces and continuous functions (=$\mathcal{T}$)
The category of groups and homomorphisms (=$\mathcal{G}$)
The category of abelian groups and homomorphisms (=$\mathcal{A}$)
The category of R-modules and homomorphisms (=$\mathcal{M}$)
The category of based topological spaces (X, x_0) and continuous functions preserving base point $(X, x_0) \to (Y, y_0)$ $(= \mathcal{T}_0)$

The category of pairs of topological spaces and pairs of continuous functions
$(X, Y) \overset{(f,g)}{\to} (X', Y') \quad (= \mathcal{T} \times \mathcal{T})$
The category of simplicial complexes and simplicial maps
The category of chain complexes and chain maps
Given (X, x_0), the category of covering spaces and morphisms

Examples of Functors

1. $F : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$

$$\begin{array}{ll} (X, Y) \mapsto X \times Y & \\ \quad \downarrow (f, g) & \downarrow (f \times g)(x \times y) = (f(x), g(y)) \\ (X', Y') \mapsto X' \times Y' & \end{array}$$

2. Forgetful functor :$\mathcal{T} \to \mathcal{S}$ and $\mathcal{G} \to \mathcal{S}$

$$\begin{array}{ll} X \mapsto \text{"}X\text{"(underlying set)} & \\ \downarrow f \quad \downarrow \text{"}f\text{"(underlying set function)} & \\ Y \mapsto \text{"}Y\text{"} & \end{array}$$

3. $\mathcal{T}_0 \xrightarrow{F=\pi_1} \mathcal{G}$

$$\begin{array}{ll} (X, x_0) \mapsto \pi_1(X, x_0) & \\ \quad \downarrow f \qquad \downarrow \pi_1(f) = f_* & \\ (Y, y_0) \mapsto \pi_1(Y, y_0) & \end{array}$$

4. Cat. simplicial cxs and simplicial maps $\overset{F}{\to}$Cat. of chain cxs and chain maps.

$$\begin{array}{ccc} K & \mapsto & \mathcal{C}(K) = \{C_p(K), \partial\} \\ \downarrow f & & \downarrow f_\sharp \\ L & \mapsto & \mathcal{C}(L) = \{C_p(L), \partial\} \end{array}$$

5. Cat. simplicial cxs and simplicial maps $\overset{H_p}{\to}$ Cat. of abel. gps and homs.

$$\begin{tikzcd}
K \arrow[rr, mapsto] \arrow[dr, "F"'] \arrow[dd, "f"'] & & H_p(K) \arrow[dd, "f_*"] \\
& \mathcal{C}(K) \arrow[ur] \arrow[dd] & \\
L \arrow[rr, mapsto] \arrow[dr] & & H_p(L) \\
& \mathcal{C}(L) \arrow[ur] &
\end{tikzcd}$$

6. Cat. of vector sps and linear trs $\xrightarrow{F}$ Cat. of vector sps and linear trs

$$\begin{array}{ccccc} V & \mapsto & V^* & \alpha \circ f \\ \downarrow f & & \uparrow f^* & \uparrow \\ W & \mapsto & W^* & \alpha \end{array}$$

This is a contravariant functor.

Natural Transformation

정의 4 $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$, two functors from a category $\mathcal{C}$ to a category $\mathcal{D}$.
A natural transformations T from F to G is a function :$Ob(\mathcal{C}) \to Mor(\mathcal{D})$

s.t. $\begin{array}{ccc} F(X) & \overset{F(f)}{\mapsto} & F(Y) \\ \downarrow T_X & \curvearrowright & \downarrow T_Y \\ G(X) & \overset{G(f)}{\mapsto} & G(Y) \end{array}$ commutes for $\forall X, Y \in Ob(\mathcal{C})$ $\qquad X \mapsto T_X$

If T_X is an equivalence, $\forall X \in Ob(\mathcal{C})$, then T is called a natural equivalence between two functors.

example: Let $(X, Y) \overset{F}{\mapsto} X \times Y$
$\qquad\qquad\qquad\qquad \overset{G}{\mapsto} Y \times X$

Then $T_{(X,Y)} : X \times Y \to Y \times X$ is a natural equivalence i.e.,
$\qquad\qquad (x, y) \mapsto (y, x)$

$$\begin{array}{lcl} F : X \times Y & \xrightarrow{f \times g} & X' \times Y' \\ \quad \downarrow T_{(X,Y)} & & \downarrow T_{(X',Y')} \\ G : Y \times X & \xrightarrow{g \times f} & Y' \times X' \end{array} \qquad \text{commutes } \forall X, Y.$$