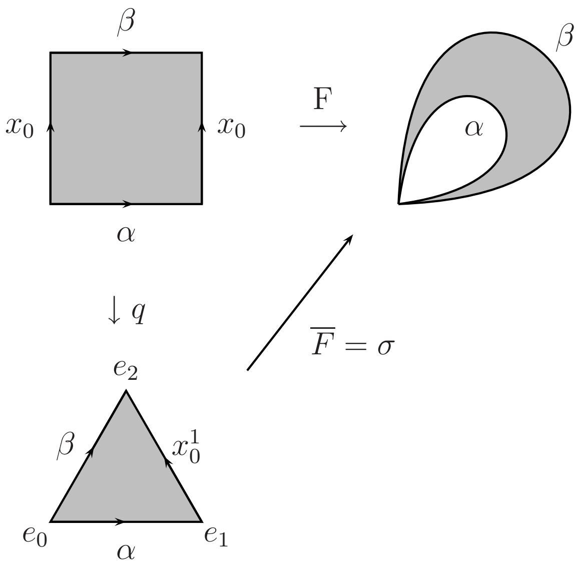
VII.5 Relation between π_1 and H_1

정리 1 *There exists a natural homomorphism* $\chi : \pi_1(X, x_0) \to H_1(X)$

$$[\alpha] \mapsto \{\alpha\}$$

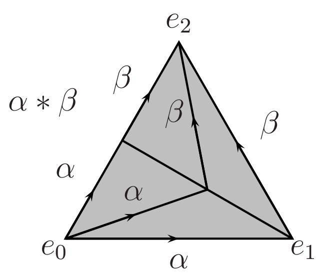
and if X *is path connected,* χ *is onto and* $ker\chi = [\pi_1, \pi_1]$ *a commutator subgroup of* π_1 *(i.e.,* $H_1 = \pi_1/[\pi_1, \pi_1]$ *is an abelianization of* π_1*.)*

증명 (1) χ is well-defined i.e., $\alpha \overset{F}{\simeq} \beta \Rightarrow \{\alpha\} = \{\beta\} (\alpha \sim \beta)$.



σ를 그림과 같이 정의하면
$\partial\sigma = x_0^1 - \beta + \alpha$
and $\partial x_0^2 = x_0^1 - x_0^1 + x_0^1 = x_0^1$
where $x_0^p : \triangle^p \to \{x_0\}$ is constant map.
따라서, $\alpha \sim \beta$.

(2) χ is a homomorphism i.e., $[\alpha * \beta] \mapsto \{\alpha\} + \{\beta\}$

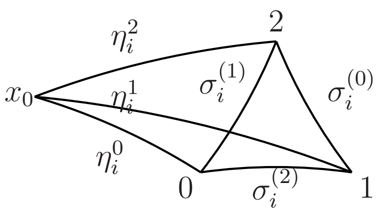


σ를 그림과 같이 정의하면 $\partial\sigma = \beta - \alpha * \beta + \alpha$이므로, $\alpha * \beta \sim \alpha + \beta$.

X가 path connected라 가정하면,
(3) χ is onto :
Given $z \in Z_1(X)$, $z = \sum n_i \alpha_i$, $\partial z = \sum n_i(\alpha_i(1) - \alpha_i(0)) = 0$.
Fix paths η_i^0 and η_i^1 from x_0 to $\alpha_i(0)$ and $\alpha_i(1)$ for each i such that $\alpha_i(0) = \alpha_j(1) \Rightarrow \eta_i^0 = \eta_j^1$.
Let $\gamma_i = \eta_i^0 * \alpha_i * \overline{\eta_i^1}$. Then,

$$\chi(\Pi[\gamma_i]^{n_i}) = \{\Sigma n_i \gamma_i\} = \{\Sigma n_i(\eta_i^0 + \alpha_i - \eta_i^1)\} = \{\Sigma n_i \alpha_i\} = \{z\}$$

(4) $ker\chi = [\pi_1, \pi_1]$.
$ker\chi \supset [\pi_1, \pi_1]$: clear since H_1 is abelian.
$ker\chi \subset [\pi_1, \pi_1]$:
Suppose $\chi[\gamma] = \{\gamma\} = 0$, i.e., $\gamma = \partial c$, $c = \sum n_i \sigma_i \in S_2(X)$.
Then, $\gamma = \partial c = \sum n_i \partial \sigma_i = \sum n_i(\sigma_i^{(0)} - \sigma_i^{(1)} + \sigma_i^{(2)})$.



Fix paths $\eta_i^0, \eta_i^1, \eta_i^2$ for each vertex of σ_i as before.
Let $\beta_i^0 = \eta_i^1 \sigma_i^{(0)} \overline{\eta_i^2}$, $\beta_i^1 = \eta_i^0 \sigma_i^{(1)} \overline{\eta_i^2}$, $\beta_i^2 = \eta_i^0 \sigma_i^{(2)} \overline{\eta_i^1}$.
Then $\beta_i := \beta_i^0 \overline{\beta_i^1} \beta_i^2 \simeq \eta_i^1 \sigma_i^{(0)} \overline{\sigma_i^{(1)}} \sigma_i^{(2)} \overline{\eta_i^1} \simeq x_0$.

Now compare γ and $\prod \beta_i^{n_i}$.
Note that $[\gamma] = [\gamma][\beta_i^{n_i}]^{-1} = [\gamma * \overline{\beta_i^{n_i}}] \in [\pi_1, \pi_1]$ by the following claim.

<u>Claim</u> Let $\delta = \prod \alpha_i^{\epsilon_i}$ (α_i: paths and $\epsilon_i = \pm 1$) be a loop.
$exp(\alpha_i) = 0 \Rightarrow [\delta] \in [\pi_1, \pi_1]$ ($exp(\alpha_i)$는 α_i의 지수 합)
<u>proof of Claim</u>
Define η_i^0, η_i^1 as before. Then,

$$\delta = \Pi \alpha_i^{\epsilon_i} \simeq \Pi(\eta_i^0 * \alpha_i * \overline{\eta_i^1})^{\epsilon_i}$$

Let $\overline{\delta}$ be the coset of $[\delta]$ in $\pi_1/[\pi_1, \pi_1]$. Then writing $\beta_i = \eta_i^0 * \alpha_i * \overline{\eta_i^1}$ we have

$$\overline{\delta} = \sum_i \epsilon_i \overline{\beta_i} = \sum_{\beta_i} exp(\alpha_i) \overline{\beta_i} = 0$$

where the last summation is over distinct β_i's (i.e., over α_i's).
Therefore, $[\delta] \in [\pi_1, \pi_1]$ $\square$

숙제 23. Compare $\pi_1(\Sigma_g)$ and $H_1(\Sigma_g)$ using the earlier computations of these.