Base point change

Let X be path-connected. For $x_0, x_1 \in X$, compare $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$. Choose a path ρ from x_0 to x_1 and define $\phi_\rho : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $[\alpha] \mapsto [\overline{\rho} * \alpha * \rho]$. Then ϕ_ρ is an isomorphism.

증명
(1) ϕ_ρ가 homomorphism임을 보이자.

$$\begin{aligned}
\phi_\rho[\alpha * \beta] &= [\overline{\rho} * \alpha * \beta * \rho] \\
&= [\overline{\rho} * \alpha * \rho * \overline{\rho} * \beta * \rho] \\
&= [\overline{\rho} * \alpha * \rho][\overline{\rho} * \beta * \rho] \\
&= \phi_\rho[\alpha]\phi_\rho[\beta].
\end{aligned}$$

(2) Define $\phi_{\overline{\rho}} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $[\beta] \mapsto [\rho * \beta * \overline{\rho}]$.
Then,$(\phi_{\overline{\rho}} \circ \phi_\rho)[\alpha] = \phi_{\overline{\rho}}[\overline{\rho} * \alpha * \rho] = [\rho * \overline{\rho} * \alpha * \rho * \overline{\rho}] = [\alpha]$.
따라서 $\phi_{\overline{\rho}} \circ \phi_\rho = id$.

Similarly, $\phi_\rho \circ \phi_{\overline{\rho}} = id$.

□

한편, 또다른 path σ가 주어졌을때 만일 $\rho \simeq \sigma$ 이라면 $\phi_\rho = \phi_\sigma$ 이다.

숙제 2.
일반적으로 homotopic 하지 않은 두 path ρ, σ에 대해

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\phi_\rho"] \arrow[dr, "\phi_\sigma"'] & \pi_1(X, x_1) \\
& \pi_1(X, x_1) \arrow[u, "\mu"']
\end{tikzcd}$$

에서 $\mu = \phi_\rho \circ \phi_{\sigma^{-1}}$ 로 두면 이는 isomorphism 이 되고, 이것은 어떤 loop에 의한 conjugation (an inner automorphism of $\pi_1(X, x_1)$)이 됨을 보여라.