I. Adjunction Space (Attaching space)

I.1 Construction

정의 1 X, Y: disjoint topological spaces , $A \overset{closed}{\subset} X$ and $f : A \to Y$, a map.
Define an equivalence relation $\sim$ on $X \coprod Y$ generated by $a \sim f(a), \quad \forall a \in A$.
The quotient space $X \underset{f}{\cup} Y = X \coprod Y / \sim$ is called the adjunction space determined by f and f is called an attaching map.

정리 1 *(Extension principle)*
Let $g : X \to Z$ and $h : Y \to Z$ s.t. $g(a) = hf(a), \quad \forall a \in A \Rightarrow$

$$\begin{tikzcd}
X \coprod Y \arrow[r, "g \coprod h"] \arrow[d, "p"] & Z \\
X \underset{f}{\cup} Y \arrow[ur, dotted] &
\end{tikzcd}$$

$\exists! k$ s.t. the diagram commute

정리 2 *Let $X \coprod Y \xrightarrow{p} X \underset{f}{\cup} Y$ be the quotient map.*
(1) Y is embedded as a closed subset of $X \underset{f}{\cup} Y$:
$p|_Y : Y \to p(Y)$ is a homeomorphism.
(2) $X - A$ is embedded as an open subset of $X \underset{f}{\cup} Y$:
$p|_{X-A} : X - A \to p(X \setminus A)$ is a homeomorphism.

증명(1) $p|_Y$ is continuous and 1-1.
Show $p|_Y$ is a closed map:
$C \subset Y$ a closed subset and show $p(C)$ is closed in $X \underset{f}{\cup} Y$,
i.e., $p^{-1}p(C) = f^{-1}(C) \coprod C$ is closed. And the assertion clearly holds.

(2) $p|_{X-A}$ is continuous and 1-1. Show it is an open map:
$U \subset X - A$ open $\Rightarrow U$ open in $X \Rightarrow p(U)$ is open since $p^{-1}p(U) = U$ is open is $X \coprod Y$. $\square$

정리 3 *(Separation Axiom)*
$X, Y : T_1 \Rightarrow X \underset{f}{\cup} Y : T_1$
X, Y : *normal* $\Rightarrow X \underset{f}{\cup} Y$: *normal*

Ref. See Munkres p.210

정의 2 (Collared pair) (X, A) is called a collared pair if
(1)$A \subset X$ is closed.
(2)X is Hausdorff.
(3) Points in $X - A$ can be separated from $A : \forall x \in X - A,\ \exists U, V$: disjoint open sets s.t. $x \in U$ and $A \subset V$.
(4) A has a collaring B in X : $\exists$ open $B \supset A$ s.t. A is a strong deformation retract of B.

명제 4 (X, A)*: a collard pair,* Y*: Hausdorff* $\Rightarrow (X \underset{f}{\cup} Y, Y)$ *: a collard pair.*
In fact, B*: a collaring of* $A \Rightarrow Y \cup p(B)$*: a collaring of* Y*.*

증명 (1) : clear from 정리 2(1)

(2) $X \underset{f}{\cup} Y$ is Hausdorff:
Case 1. $z_1, z_2 \in X \underset{f}{\cup} Y - Y \cong X - A \Rightarrow$clear.

Case 2. $z_1 \in Y, z_2 \notin Y \overset{\text{정의 } 2(3)}{\Rightarrow} \exists U \ni z_2, V \supset A$
$\Rightarrow p(U)$: open neighborhood of z_2 and $p(V) \cup Y$: open neighborhood of z_1 gives a separation. (Note $p^{-1}(p(V) \cup Y) = V \coprod Y$: open in $X \coprod Y$.)

Case 3. $z_1, z_2 \in Y$: Let $z_1 \in V_1, z_2 \in V_2$ be a separation and
$r : B \to A$ a strong deformation retract. Let $U_i = r^{-1} f^{-1}(V_i)$: open in X.
$\Rightarrow p(U_1) \cup p(V_1)$ and $p(U_2) \cup p(V_2)$ give a separation for z_1 and z_2
(Note $p^{-1}(p(U) \cup p(V)) = U \coprod V$.)

(3) $z \notin Y$. Then use 정의 2(3) to get disjoint open sets $U \ni z$ and $V \supset A \Rightarrow$ $p(U)$ and $p(V) \cup Y$ give a separation for z and Y.(cf. Case2.)

(4) Let $D : id \simeq i \cdot r(\text{rel } A)$ be a strong deformation retract :

$$D : B \times I \to B \text{ s.t. } \left\{ \begin{array}{cc} D(a, t) = a, & \forall a \in A \quad t \in I \\ D(b, 0) = b, & \forall b \in B \\ D(b, 1) = r(b) \in A, & \forall b \in B \end{array} \right\}$$

Define $\bar{D} : p(B) \cup Y \times I \to p(B) \cup Y$ by $\bar{D}(z,t) = \left\{ \begin{array}{cc} z, & z \in Y \\ p(D(b,t)), & z = p(b), \quad b \in B - A \end{array} \right\}$

$$\begin{tikzcd} (B \coprod Y) \times I \arrow[r, "D \coprod p_1"] \arrow[d, "p \times id"] \arrow[dr, phantom, "\curvearrowright"] & B \coprod Y \arrow[d, "p"] \\ (p(B) \cup Y) \times I \arrow[r, "\bar{D}"] & p(B) \cup Y \end{tikzcd}$$

$\Rightarrow$ $\bar{D}$ is continuous by the following fact.

Fact. $p : X \to Y$ quotient , C : locally compact Hausdorff.
$\Rightarrow p \times id : X \times C \to Y \times C$ is a quotient map.
증명 Ref. Munkres p.113 $\square$ $\square$