# III.2 $H_q(M^n, M-A), \quad q \geqslant n$

idea : Compare $H_n(M, M-A)$ with $\Gamma A$.

6. Suppose $U^{open} \subset M$. Then $\beta_U \in H_n(M, M-U)$ can be viewed as a section as before and we have a homomorphism $j_U : H_n(M, M-U) \to \Gamma U$.

$$\beta_U \mapsto j_U(\beta_U) : x \mapsto \beta_U|_x$$

In general, $\forall A \subset M$, does $j_A : H_n(M, M-A) \to \Gamma A$(defined by "$\beta$" $= j_A(\beta) : x \mapsto \beta|_x$) define a homomorphism, i.e., is $j_A(\beta)$ continuous section on $A$?

**증명** Want "$\beta$" is locally constant, i.e., $\forall a \in A$, $\exists V$ and $\beta_V$ s.t. $\beta|_x = \beta_V|_x$ , $\forall x \in A \cap V$.
Recall : Can represent $\beta = \{b\}$ with $\partial b \subset M - A$.
$|\partial b|$ : compact $\Rightarrow U = M - |\partial b|$ is open and choose $V$, a coordinate ball neighborhood of $a$ with $V \subset U$.

$$\begin{array}{ccc} (M, M-U) & \to & (M, M-V) \\ \downarrow & & \downarrow \cong on H_n \\ (M, M-A) & \to & (M, M-a) \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \beta' = \{b\} & \dashrightarrow & \exists \beta_V \\ \downarrow & & \updownarrow \cong \\ \beta'|_A = \beta & \to & \beta|_a \end{array}$$

And $\forall x \in A \cap V$, $\beta_V|_x = \beta'|_V|_x = \beta'|_x = \beta'|_A|_x = \beta|_x$. □

7. When is $j_A : H_n(M, M-A) \to \Gamma A$ an isomorphism?
Know : true if $A = U$, a coordinate ball.

$$\begin{array}{ccccc} H_n(V, V-U) & \xrightarrow{excision:\cong} & H_n(M, M-U) & \xrightarrow{j_U} & \Gamma U \\ \downarrow \psi_*:\cong & & & & \downarrow \psi_*:\cong \\ \mathbb{Z} = H_n(\mathbb{R}^n, \mathbb{R}^n \backslash D) & & \xrightarrow{j_D:\cong} & & \Gamma D = \mathbb{Z} \end{array}$$

Also note $H_q(M, M-U) \cong H_q(\mathbb{R}^n, \mathbb{R}^n \backslash D) \cong \widetilde{H_{q-1}}(S^{n-1}) = 0$ if $q > n$.

Let $M = \mathbb{R}^n$.
If $A$ is a "nice " compact set, then $j_A$ is $\cong$.
e.g. $A = D, [0,1] \times [0,1], [0,1]$, point, $\cdots$ etc.
But note that if $A = M = \mathbb{R}^1$, $H_1(\mathbb{R}^1, \mathbb{R}^1 \backslash \mathbb{R}^1) = H_1(\mathbb{R}^1) = 0$ but $\Gamma \mathbb{R}^1 = \mathbb{Z}$.

**Note.** $A$: closed. Then

$$\begin{array}{ccc} H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \\ & \searrow^{j_A} \quad \nearrow_{\subset} & \\ & \Gamma_c A & \end{array}$$

i.e., $j_A(\beta) \in \Gamma_c A$ , $\forall \beta$ where $\Gamma_c A$ consists of sections with compact support.

**증명**Let $\beta = \{b\}$: relative cycle $\Rightarrow |b|$ : compact.
Then $\forall x \in (M - |b|) \cap A, \beta|_x = 0$ since $|b| \subset M - x$. Think of this in chain level.

$$\begin{array}{ccccccccc} 0 & \to & S_n(M-A) & \to & S_n(M) & \to & S_n(M)/S_n(M-A) & \to & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow & & \\ 0 & \to & S_n(M-x) & \to & S_n(M) & \to & S_n(M)/S_n(M-x) & \to & 0 \end{array}$$

$\therefore \beta$ has a support $\subset |b| \cap A$ : compact □

So the right statement is $j_A : H_n(M, M-A) \to \Gamma_c A$.
Furthermore, $j_A$ is natural:

$$B \subset A \subset M \Rightarrow \qquad \begin{array}{ccc} H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \\ \downarrow{\rho_B^A} & & \downarrow{restriction} \\ H_n(M, M-B) & \xrightarrow{j_B} & \Gamma B \end{array} \qquad \text{commute.}$$

Exercise. **숙제 11.**

$$f : (M, A) \xrightarrow{\cong} (N, B) \Rightarrow \qquad \begin{array}{ccc} H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \\ \downarrow{f_*:\cong} & & \downarrow{\text{"}f_*\text{"}} \\ H_n(N, N-B) & \xrightarrow{j_B} & \Gamma B \end{array} \qquad \text{commute.}$$

**8. (Theorem)** Let $M$ be an $n$-dimensional manifold and $A^{\text{closed}} \subset M$. Then
(1) $H_q(M, M-A) = 0$ for $q > n$
(2) $H_n(M, M-A) \cong \Gamma_c A$

**증명**

**보조정리 1** *(MV) $A, B$ closed $\subset M$. If the theorem is true for $A, B$ and $A \cap B$, then so is for $A \cup B$.*

**증명** $A, B$ : closed $\Rightarrow M-A, M-B$ open with $(M-A)\cap(M-B) = M-A\cup B$
$(M-A)\cup(M-B) = M-A\cap B$

relative MV:

$$\begin{array}{ccccccccccc}
\text{hypothesis} \to & 0 & \to & H_n(M, M-A\cup B) & \to & H_n(M, M-A)\oplus H_n(M, M-B) & \to & H_n(M, M-A\cap B) & \to & 0 \\
& \downarrow\cong & & \downarrow j & & \downarrow\cong & & \downarrow\cong & & \\
\text{exact :} & 0 & \longrightarrow & \Gamma(A\cup B) & \longrightarrow & \Gamma(A)\oplus\Gamma(B) & \longrightarrow & \Gamma(A\cap B) & \longrightarrow & 0
\end{array}$$

$$s \longmapsto (s|_A, s|_B)$$

$$(a, b) \longmapsto a|_{A\cap B} - b|_{A\cap B}$$

(i) follows from relative MV-sequence.
(ii) follows from 5-lemma. □

**보조정리 2** *If $M = \mathbb{R}^n$ and $A$ is a compact subset of $\mathbb{R}^n$, then the theorem is true.*

**증명** Know : The theorem is true for a "nice" compact set $A \subset \mathbb{R}^n$, for instance $A$ =rectangle.
By lemma 1, the theorem is true if $A$ is a finite union of rectangles by induction on number of rectangles. □

이제 compact set $A$에 대하여 정리가 성립함을 보이자.
먼저 $j_A : H_n(M, M-A) \to \Gamma A = \Gamma_c A$가 onto임을 보인다.
(i) For $s \in \Gamma A$, there exists an open set $U$ containing $A$ such that $s$ can be extended to $\bar{s}$ on $U$.
pf) $s(A)$ is compact $\Rightarrow s(A)$ lies in finitely many sheets of $\mathbb{R}^n_{\mathcal{O}} \cong \mathbb{R}^n \times \mathbb{Z}$.
Let $A_i$=$s^{-1}(\mathbb{R}^n \times i)$ for $i \in \mathbb{Z}$. $A_i$ is compact.
So there exists an open set $U_i$ containing $A_i$ such that $U_i$'s are pairwise disjoint and $s$ can be extended to $\bar{s}$ over $U = \bigcup U_i$.

(ii) Cover $A$ by finitely many rectangles in $U$ and let $A'$ be the union of rectangles.

Then the following diagram commutes.

$$\begin{array}{ccc} H_n(M, M-A') & \xrightarrow[\cong]{j_{A'}} & \Gamma A' \ni \bar{s}|_{A'} \\ \downarrow & & \downarrow \text{restriction} \\ H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \ni s \end{array}$$

임의의 $s \in \Gamma A$에 대하여 (i)에 의하여 $\bar{s}$가 존재하므로 $\bar{s}|_{A'}$은 $s$에 대응되는 $\Gamma A'$의 원소이다. 또한 $j_{A'}$은 isomorphism이므로 $j_A$는 onto이다.

이제 $j_A$이 1-1이고 $H_q(M, M-A) = 0$ for $q > n$임을 보이자.
Let $\alpha \in H_q(M, M-A)$ and assume $j_A(\alpha) = 0$ if $q = n$.
Suppose $\alpha = \{a\}$ with $\partial a \subset M - A$. Since $|\partial a|$ is compact, $V = M - |\partial a|$ is open. Let $\alpha' = \{a\} \in H_q(M, M-V)$.

$(q = n)$ : Let $U \subset V$ be an open set containing $A$ each of whose components intersects $A$. Let $A'$ be a finite union of rectangles which covers $A$ and is contained in $U$.
Then $j_U(\alpha'|_U)$ is a section on $U$ which has a zero on each of its component.
($\because$ Each component intersects $A$ and $j_A(\alpha) = 0$.)
By uniqueness of sections on connected sets, $j_U(\alpha'|_U) = 0$.

$$\begin{array}{ccc} H_n(M, M-U) & \xrightarrow{j_U} & \Gamma U \\ \downarrow & & \downarrow \\ H_n(M, M-A') & \xrightarrow[\cong]{j_{A'}} & \Gamma A' \\ \downarrow & & \downarrow \\ H_n(M, M-A) & \xrightarrow{j_A} & \Gamma A \end{array} \qquad \begin{array}{ccc} \alpha'|_U & \longmapsto & 0 \\ \downarrow & & \downarrow \\ \alpha'|_{A'} & \longmapsto & 0 \\ \downarrow & & \downarrow \\ \alpha & \longmapsto & 0 \end{array}$$

Therefore by the above diagram, $\alpha'|_{A'} = 0$ and hence $\alpha = 0$.
$(q > n)$ : The theorem is true for $A'$. So $\alpha'|_{A'} = 0$ and hence $\alpha = 0$. □

**정리의 증명**
**Step 1.** $A^{\text{compact}} \subset M$:
$A$ is a finite union of compact sets each of which is contained in a coordinate ball neighborhood($\approx \mathbb{R}^n$) and apply lemma 1 and lemma 2.

**Step 2.** $A \subset U^{\text{open}} \subset \overline{U}^{\text{compact}} \Rightarrow$ The theorem is true for $A \subset U(= M)$[1] where $A$ is a closed subset of $U$:

[1] manifold의 open subset은 manifold이므로 $U = M$으로 하여도 무방하다.

$\partial U = \overline{U} - U$ is compact and $U - \overline{A} = U - A$.
Consider $(M, M - \partial U, M - (\partial U \cup \overline{A}))$.
$(q > n)$:

$$H_{q+1}(M, M-\partial U) \to H_q(M-\partial U, M-(\partial U \cup \overline{A})) \to H_q(M, M-(\partial U \cup \overline{A})) \to \cdots$$

By excision theorem, $H_q(M - \partial U, M - (\partial U \cup \overline{A})) = H_q(U, U - A)$.
By step 1, $H_{q+1}(M, M - \partial U) = 0$ and $H_q(M, M - (\partial U \cup \overline{A})) = 0$.
Therefore $H_q(U, U - A) = 0$.
$(q = n)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(U, U-A) & \longrightarrow & H_n(M, M-(\partial U \cup \overline{A})) & \longrightarrow & H_n(M-\partial U) \\ & & \downarrow{\scriptstyle j_A} & & \downarrow{\scriptstyle \cong \text{(step 1)}} & & \downarrow{\scriptstyle \cong \text{(step 1)}} \\ 0 & \longrightarrow & \Gamma_c A & \xrightarrow[\substack{\text{extension by 0} \\ \text{outside support}}]{} & \Gamma(\partial U \cup \overline{A}) & \xrightarrow[\text{restriction}]{} & \Gamma(\partial U) \end{array}$$

So, $j_A$ is an isomorphism by 5 lemma.

**Step 3.**(general case)
Show $j_A : H_n(M, M - A) \to \Gamma_c A$ is onto:
$\forall s \in \Gamma_c A$, let supp $s = K$.
Then K is compact and $\exists U$ such that $K \subset U \subset \overline{U}^{\text{compact}}$
Let $A' = A \cap U$.

$$\begin{array}{ccc} H_n(U, U-A') & \xrightarrow{\cong \text{(step 2)}} & \Gamma_c A' \ni s|_{A'} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \substack{\text{extension by 0} \\ \text{outside } K}} \\ H_n(M, M-A) & \xrightarrow{j_A} & \Gamma_c A \ni s \end{array}$$

So, $j_A$ is onto.

Show $j_A$ is 1-1 and $H_q(M, M - A) = 0$ if $q > n$:
$\alpha \in H_q(M, M - A)$ and assume $j_A(\alpha) = 0$ if $q = n$.
If $\alpha = \{a\}$ with $\partial a \subset M - A$, $|a|$: compact $\Rightarrow |a| \subset U \subset \overline{U}^{\text{compact}}$.
Let $A' = A \cap U$. Apply the above diagram.
$q = n$ : Since $|a| \subset U$, $\alpha' = \{a\} \in H_n(U, U - A')$. So in the above diagram, $i_*(\alpha') = \alpha$. Since $j_A(\alpha) = 0$ and $0|_{A'} = 0$, $j_{A'}(\alpha') = 0$. So $\alpha' = 0$ and hence $\alpha = 0$.
$q > n$ : By step 2, $\alpha' = \{a\} \in H_q(U, U - A') = 0$. So $\alpha = 0$. □

**9. Consequences of the theorem**

(1) Let $A$ be connected and closed but not compact. Then $H_n(M, M-A) = 0$. In particular if $M$ is connected but not compact then $H_n(M) = 0$.

**증명** If $s \in \Gamma_c A$, $\nu \circ s : A \to \mathbb{Z}^{\geq 0}$ is continuous and $= 0$ at some $a \notin \text{supp } s$. So $\Gamma_c A = 0$ □

(2) If $M$ is orientable along $A$, $A$ is compact and has $k$ components, then $H_n(M, M-A) \cong \mathbb{Z}^k$.

**증명** $\Gamma_c A = \Gamma A \cong \mathbb{Z}^k$ by 5. (2). □

(3) If $A \subset \mathbb{R}^n$ is compact and has $k$ components, then $\widetilde{H}_{n-1}(\mathbb{R}^n - A) \cong \mathbb{Z}^k$.

**증명** From homology sequence of pair $(\mathbb{R}^n, \mathbb{R}^n - A)$, we get

$$0 \to H_n(\mathbb{R}^n, \mathbb{R}^n - A) \to \widetilde{H}_{n-1}(\mathbb{R}^n - A) \to 0$$

So $\widetilde{H}_{n-1}(\mathbb{R}^n - A) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - A) \cong \mathbb{Z}^k$ by (2). □

(4) If $M$ is connected and closed[2], then $H_n(M) = \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable} \\ 0, & \text{if } M \text{ is non-orientable} \end{cases}$

**증명** Clear from 5. (3) and (4). □

**Remark.** For a PID $R$, $H_n(M) = \begin{cases} R, & \text{if } M \text{ is } R-\text{orientable} \\ 0, & \text{if } M \text{ is not } R-\text{orientable} \end{cases}$

**10. Fundamental class of $M$ and degree**

A choice of generating section is an orientation and the corresponding homology class $\zeta$[3] is called the ***fundamental (orientation) class*** of $M$, i.e., $\zeta|_x \in H_n(M, M-x)$ is the preferred orientation at $x$ for all $x \in M$.

Let $M^n, N^n$ be oriented closed connected manifolds. For $f : M \to N$, if $f_*(\zeta_M) = k \cdot \zeta_N$, $k$ is called the ***degree*** of $f$.

**숙제 14** $\forall y \in N$, "regular value" i.e., $\forall x \in f^{-1}(y)$, $f$ is a homeomorphism on a neighborhood $U_x$ $(f|_{U_x} : U_x \xrightarrow{\cong} V_y)$. Then

$$deg f = \sum_{x \in f^{-1}(y)} deg_x f$$

[2] A manifold $M$ is closed if it is compact without boundary.

[3] $[M]$으로 쓰기도 한다.

where $deg_x f$ is defined by $f|_{U*} : H_n(U, U - x) \to H_n(V, V - y)$.
In particular, if $p : M \to N$ is a $k$-fold covering, $degp = k$ (with respect to the induced orientation on M).

**숙제 15** (22.43) (22.49) (22.50)*