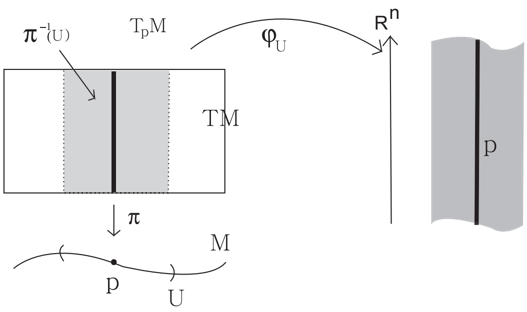
Vector Bundle.

예 1 Typical example : Tangent bundle.
$TM = \bigcup_p T_pM$: tangent bundle.
$\downarrow \pi$
M
For each p$\in M$, T_pM is an n-dim'l vector space and these are pieced together smoothly to give a manifold TM.
i.e. $\forall p \in M, \exists U$ a neighborhood of p and a diffeomorphism

$$\begin{aligned} \varphi_U : \pi^{-1}(U) &\longrightarrow U \times \mathbb{R}^n \\ X_p &\mapsto (p, a^1, \cdots, a^n) \\ \text{where } X_p = \textstyle\sum_{i=1}^n a^i \frac{\partial}{\partial x_i}|_p &\text{ and } a^i = dx^i(X_p) = X_p(x^i) \end{aligned}$$

s.t. $\varphi_U|_p : \pi^{-1}(p) = T_pM \to \{p\} \times \mathbb{R}^n$ is a vector space isomorphism.

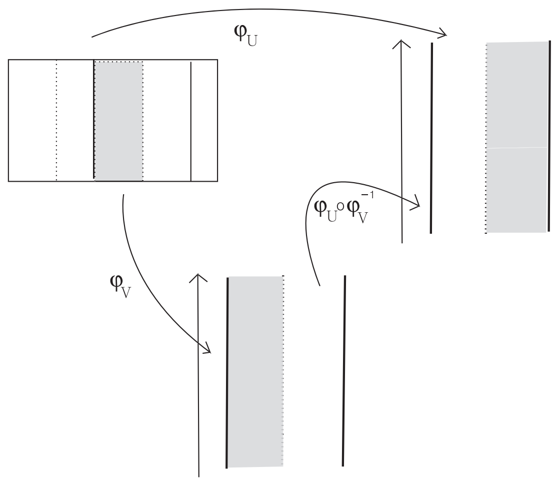


정의 1 A n-dim'l vector space over M is a triple (E,π,M), where $\pi : E \to M$ is a $\mathcal{C}^\infty$-map between $\mathcal{C}^\infty$-manifolds toghther with a vector space (over $\mathbb{R}$ or $\mathbb{C}$) structure on each fiber $\pi^{-1}(p)$ s.t. the following local triviality condition is satisfied:
$\forall p \in M$,$\exists U$ a neighborhood of p and a diffeomorphism

$$\begin{array}{ccc} \varphi_U : \pi^{-1}(U) = E_U & \longrightarrow & U \times \mathbb{R}^n \\ \pi \searrow & & \swarrow p \\ & U & \end{array}$$

s.t. $\varphi_U|_p : \pi^{-1}(p) \longrightarrow \{p\} \times \mathbb{R}^n$ is a vector space isomorphism $\forall p \in U$.

transition map
Given two local trivilizations φ_U,φ_V, the map $g_{UV}(p) : U \cap V \longrightarrow GL(n, \mathbb{R})$ given by $g_{UV}(p) = \varphi_U \circ {\varphi_V}^{-1}|_{p\times\mathbb{R}^n}$ is called a transition function relative to φ_U and φ_V.



TM case :

$$X_p = \sum a^i \frac{\partial}{\partial x_i} = \sum b^j \frac{\partial}{\partial y_j}$$

$$\begin{array}{ccc} \downarrow & & \searrow \\ (p, a^1, \cdots, a^n) & \overset{g_{UV}(p)}{\longleftarrow} & (p, b^1, \cdots, b^n) \end{array}$$

에서 $g_{UV}(p)(b) = a$, 여기서 a와 b의 관계를 구하면 $a^i = dx^i(X_p) = \sum \frac{\partial x_i}{\partial y_j} dy^j(X_p) = \sum \frac{\partial x_i}{\partial y_j} b^j$ 이다.
$\therefore g_{UV}(p) = \frac{\partial x}{\partial y}(p) \leftarrow n \times n$ Jacobian matrix.

Claim 1 *Given φ_U and φ_V without condition $\mathcal{C}^\infty$. Show that g_{UV} is $\mathcal{C}^\infty$ as a map from $U \cap V$ to $GL(n, \mathbb{R}) \Leftrightarrow \varphi_U \circ \varphi_V^{-1}$ is $\mathcal{C}^\infty$.* **(숙제#18)**