Effect of Mappings

$\varphi : M \to N,\ \mathcal{C}^\infty \ \Rightarrow\ \varphi_* : TM \to TN$ *bundle map.*
And $\varphi_*|_p : T_pM \to T_{\varphi(p)}N$, *linear* $\Rightarrow (\varphi_*|_p)^* : T^*_{\varphi(p)}N \to T^*_pM$, dual map.
This induces a linear map $\varphi^*_p := \bigwedge^r(\varphi_*|^*_p) : \bigwedge^r(T^*_{\varphi(p)}N) \to \bigwedge^r(T^*_pM)$

For $\alpha \in \mathcal{E}^1(N)$, $\varphi^*\alpha \in \mathcal{E}^1(M)$ *is defined as* $\varphi^*\alpha|_p = \varphi^*_p(\alpha|_{\varphi(p)})$.
Note that $\varphi^*\alpha(X_p) = \alpha(\varphi_* X_p)$

Similarly, for $\alpha \in \mathcal{E}^r(N)$, $\varphi^*\alpha \in \mathcal{E}^r(M)$ and

$$\begin{aligned}\varphi^*\alpha(X_1, \cdots, X_r)(p) &= \varphi^*\alpha(X_1(p), \cdots, X_r(p)) \\ &= \alpha((\varphi_*X_1)(\varphi(p)), \cdots, (\varphi_*X_r)(\varphi(p))) \\ &= \alpha(\varphi_*X_1, \cdots, \varphi_*X_r)(\varphi(p))\end{aligned}$$

$\Rightarrow\ \varphi^* : \mathcal{E}(N) \to \mathcal{E}(M) = \bigoplus_{r=0}^{\infty} \mathcal{E}^r(M).$

이 경우 differential d에 관하여 $\mathcal{E}(M)$은 "differential graded algebra"가 된다.

명제 1 *Let* $\varphi : M \to N$, $\mathcal{C}^\infty$. *Then* $\varphi^* : \mathcal{E}(N) \to \mathcal{E}(N)$ *is a differential graded algebra homomorphism, i.e.,*

1. $\varphi^*(\alpha + \beta) = \varphi^*\alpha + \varphi^*\beta$
2. $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$
3. $\varphi^* d = d\varphi^*$

증명 1 and 2 are clear.
$(\because)\ f : V \to W \Rightarrow f^* : W^* \to V^*$
$\Rightarrow \bigwedge(f^*) : \bigwedge(W^*) \to \bigwedge(V^*)$ is an algebra homomorphism.
And note that $+, \wedge$ are pointwise operations.

Show $\varphi^*\alpha$ is $\mathcal{C}^\infty$ if $\alpha \in \mathcal{E}^r(N)$:
1st $\varphi^*(df)$ case :
$\varphi^*(df)(X)(p) = \varphi^*(df)(X_p) = df(\varphi_*X_p)$
$\qquad = (\varphi_*X_p)f = X_p(f \circ \varphi) = X(f \circ \varphi)(p) \ : \mathcal{C}^\infty - function\ of\ p.$
Moreover, $\qquad = d(f \circ \varphi)(X)(p)$
$\therefore\ \varphi^*(df) = d(f \circ \varphi) = d(\varphi^* f)$ and $\varphi^* d = d\varphi^*$ *on* $\mathcal{F}$

$2nd\ \ \alpha \in \mathcal{E}^r(N)$ case:

Locally $\varphi \ : V \to (U, x)$

$\qquad\qquad p \mapsto \varphi(p)$

Let $\alpha = \sum f_I dx_I$. Then $\varphi^*\alpha = \sum \varphi^*(f_I)\varphi^*(dx_{i_1}) \wedge \cdots \wedge \varphi^*(dx_{i_r}) \quad : \mathcal{C}^\infty$.

Now we can prove 3 as follows.

$d(\varphi^*\alpha) = d(\sum(\varphi^* f_I) d(\varphi^* x_{i_1}) \wedge \cdots \wedge d(\varphi^* x_{i_r}))$

$= \sum d(\varphi^* f_I) \wedge d(\varphi^* x_{i_1}) \wedge \cdots \wedge d(\varphi^* x_{i_r})$

$= \varphi^*(\sum df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = \varphi^* d\alpha$ $\square$