$\mathcal{L}_X$ and i_X

· i_X : 임의의 $X \in \mathcal{X}(M)$에 대해 $i_X : \mathcal{E}^p \to \mathcal{E}^{p-1}$는 다음과 같이 정의된다.

$i_X\alpha|_p := i_{X_p}(\alpha_p) \qquad \forall p \in M$(pointwise operation)
or equivalently, $i_X\alpha(X_2, \cdots, X_p) = \alpha(X, X_2, \cdots, X_p)$

Recall : i_X is an antiderivation of deg -1
i.e. $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X\beta)$ and $i_X^2 = 0$

· $\mathcal{L}_X$: 주어진 tensor field K에 대해 X 방향으로의 Lie derivative $\mathcal{L}_X$ 은 다음과 같이 정의된다.

$(\mathcal{L}_X K)_p := \lim_{t\to 0} \frac{1}{t}((\varphi_t^* K)_p - K_p)$
where $\{\varphi_t\}$ is a flow of X and $\varphi_t^* Y := \varphi_{t*}^{-1} Y$.

Recall : $\mathcal{L}_X f = Xf$ and $\mathcal{L}_X Y = [X, Y]$

명제 1 *(a)* $\mathcal{L}_X : \mathcal{T} \to \mathcal{T}$ *is a tensor derivation,*
i.e. $\mathcal{L}_X(K \otimes L) = (\mathcal{L}_X K) \otimes L + K \otimes (\mathcal{L}_X L)$.

(b) $\forall \alpha \in \mathcal{E}^1$ *and* $\forall Y \in \mathcal{X}, \qquad \mathcal{L}_X(\alpha(Y)) = (\mathcal{L}_X\alpha)Y + \alpha(\mathcal{L}_X Y)$.

(c) $\mathcal{L}_X : \mathcal{E}(M) \to \mathcal{E}(M)$ *is a derivation, i.e.*
$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta)$.

(d) (Cartan formula) $\mathcal{L}_X = i_X \circ d + d \circ i_X$ *on* $\mathcal{E}(M)$.

증명 (a) and (c) 보통 미분의 곱에 대한 Leifniz rule 증명과 같다.

(b) $$\begin{aligned}\mathcal{L}_X(\alpha(Y)) &= \lim_{t\to 0}\frac{1}{t}(\varphi_t^*(\alpha(Y)) - \alpha(Y)) \\ &= \lim_{t\to 0}\frac{1}{t}((\varphi_t^*\alpha)(\varphi_t^*Y) - \alpha(\varphi_t^*Y) + \alpha(\varphi_t^*Y) - \alpha(Y)) \\ &= \lim_{t\to 0}\frac{1}{t}((\varphi_t^*\alpha)(\varphi_t^*Y) - \alpha(\varphi_t^*Y)) + \lim_{t\to 0}\frac{1}{t}(\alpha(\varphi_t^*Y) - \alpha(Y)) \\ & \qquad\quad \downarrow(\text{by 아래각주}^1) \qquad\qquad \downarrow(\text{by 아래각주}^2) \\ &= \qquad (\mathcal{L}_X\alpha)(Y) \qquad\quad + \qquad \alpha(\mathcal{L}_XY)\end{aligned}$$

(d) It suffices to check on $\mathcal{F}$ and $\mathcal{E}^1$, since a derivation is a local operator and is completely determined by its action on $\mathcal{F}$ and $\mathcal{E}^1$.

$\mathcal{L}_Xf = (i_X\circ d + d\circ i_X)f \quad , \forall f\in\mathcal{E}^1$:
$\because (i_X\circ d + d\circ i_X)f = i_X(df) + 0 = df(X) = Xf = \mathcal{L}_Xf \quad , \forall f\in\mathcal{E}^1$

$\forall\alpha\in\mathcal{E}^1$and $\forall Y\in\mathcal{X}$,

$$(\mathcal{L}_X\alpha)Y = \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_XY) = X(\alpha(Y)) - \alpha([X,Y]).$$

$$\begin{aligned}(i_X\circ d + d\circ i_X)(\alpha)(Y) &= i_X(d\alpha)(Y) + d(i_X(\alpha))(Y) \\ &= d\alpha(X,Y) + d(\alpha(X))Y \\ &= d\alpha(X,Y) + Y(\alpha(X)) \\ &= X(\alpha(Y)) - \alpha([X,Y]).\end{aligned}$$

$$\therefore (\mathcal{L}_X\alpha)Y = (i_X\circ d + d\circ i_X)(\alpha)(Y).$$

□

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[1] $\lim_{t\to 0}\frac{1}{t}(\varphi_t^*\alpha(\varphi_t^*Y) - \alpha(\varphi_t^*Y)) = \lim_{t\to 0}\frac{1}{t}(\varphi_t^*\alpha - \alpha)(\varphi_t^*Y) = (\mathcal{L}_X\alpha)(Y)$

[2] $\lim_{t\to 0}\frac{1}{t}(\alpha(\varphi_t^*Y) - \alpha(Y)) = \lim_{t\to 0}\frac{1}{t}\alpha(\varphi_t^*Y - Y) = \lim_{t\to 0}\alpha(\frac{1}{t}(\varphi_t^*Y - Y))$
$= \alpha(\lim_{t\to 0}\frac{1}{t}(\varphi_t^*Y - Y)) = \alpha(\mathcal{L}_XY)$

따름정리 2 *$\mathcal{L}_X$ is a total derivation, i.e.,*
$\mathcal{L}_X(\alpha(X_1, \cdots, X_p)) = (\mathcal{L}_X\alpha)(X_1, \cdots, X_p) + \alpha(\mathcal{L}_X X_1, X_2, \cdots, X_p) + \cdots + \alpha(X_1, \cdots, X_{p-1}, \mathcal{L}_X X_p)$
$, \forall \alpha \in \mathcal{E}^p$ *and* $X_i \in \mathcal{X}$.

증명 It suffices to show for $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_p, \quad \alpha_i \in \mathcal{E}^1$.

$\mathcal{L}_X((\alpha_1 \otimes \cdots \otimes \alpha_p)(X_1, \cdots, X_p))$
$= \mathcal{L}_X(\alpha_1(X_1) \cdots \alpha_p(X_p))$
$= \sum_i \alpha_1(X_1) \cdots \mathcal{L}_X(\alpha_i(X_i)) \cdots \alpha_p(X_p)$
$\downarrow$ (by 아래각주[3])
$= \sum_i (\alpha_1 \otimes \cdots \otimes \mathcal{L}_X \alpha_i \otimes \cdots \otimes \alpha_p)(X_1, \cdots, X_p) + \sum_i (\alpha_1 \otimes \cdots \otimes \alpha_p)(X_1, \cdots, \mathcal{L}_X X_i, \cdots, X_p)$
$= \mathcal{L}_X(\alpha_1 \otimes \cdots \otimes \alpha_p)(X_1, \cdots, X_p) + \sum_i \alpha(X_1, \cdots, \mathcal{L}_X X_i, \cdots, X_p)$

□

숙제 20. Prove invariant formula of $d\omega$ using the above corollary and Cartan formula. (Hint : Use induction.)

[3] $\mathcal{L}_X(\alpha_i(X_i)) = (\mathcal{L}_X(\alpha_i))(X_i) + \alpha_i(\mathcal{L}_X X_i)$