I.1 Metric Space

Definition 1 A set X is a metric space if there exists a function (called metric) $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$; $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Examples The followings are metric spaces.

1. $\mathbb{R}^n$ with $d(x, y) = \|x - y\|$
2. (S^2, d) : the unit sphere with the spherical distance d along the geodesics.
3. A space X with a metric d defined by
$$d(x, y) = 0 \text{ if } x = y$$
$$d(x, y) = 1 \text{ if } x \neq y.$$
4. A normed vector space V with a metric $d(x, y) = \|x - y\|$.
5. Let V be a (real or complex) vector space.
A norm $\| \quad \|$ on V is a function $\| \quad \| : V \to \mathbb{R}$ satisfying the following properties :
(1) $\|x\| = 0 \iff x = 0$
(2) $\|\lambda x\| = |\lambda|\|x\|, \quad \forall \lambda \in F (=\mathbb{R} \text{ or } \mathbb{C})$
(3) $\|x + y\| \leq \|x\| + \|y\|$
6. Note that if $(V, <,>)$ is a vector space with an inner product, then $\|x\| =< x, x >^{\frac{1}{2}}$ defines the induced norm on V :
Use CS-inequality for (3) and CS-inequality follows from the inequality,
$$0 \leq \|tx + y\|^2 =< tx + y, tx + y >= \|x\|^2 t^2 + 2 < x, y > t + \|y\|^2$$
7. Let $\mathcal{B}$ be the set of all bounded function of $[a, b] \subset \mathbb{R}$. Then $\mathcal{B}$ is a vector space.
Define
$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

Then it is easy to show that $\|f+g\| \leq \|f\| + \|g\|$

$$(\sup_{x\in[a,b]} |f(x)+g(x)| \leq \sup_{x\in[a,b]} (|f(x)|+|g(x)|) \leq \sup_{x\in[a,b]} |f| + \sup_{x\in[a,b]} |g|)$$

8. Let $\mathcal{C}[a,b]$ be the set of all continuous functions an $[a,b]$

$$\|f\|_1 := \int_a^b |f|,$$

$$\|f\|_2 := \left(\int_a^b |f|^2\right)^{\frac{1}{2}}$$

defines respectively a norm on the vector space $\mathcal{C}[a,b]$.
Note that $\| \quad \|_2$ is the induced norm from an inner product defined by $< f, g >= \int fg$

Note 1. A subspace S of a metric space (X,d) inherites a metric d from X
2. A product of two metric spaces, (X_1,d_1) and (X_2,d_2), admits a product metric d on $X = X_1 \times X_2$.

Homework 1 Prove the followings

1. Let X_1 and X_2 be metric spaces with metric d_1 and d_2, respectively. Then $X = X_1 \times X_2$ is also a metric space with metric d given by,

$$d(x,y) = \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2}$$

for all $x = (x_1,y_1), y = (x_2,y_2)$

2. Let X_i be a metric space with a metric d_i. Define $d_i'(x_i,y_i) = min\{d_i(x_i,y_i),1\}$. Then $X = \prod_{i=1}^{\infty} X_i$ is a metric space with a metric d given by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{d_i'(x_i,y_i)}{2^i}$$

3. Now define $d_i'(x_i,y_i) = \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$ and prove the same.

Open sets

Definition 2 Let (X, d) be a metric space. Then U is an open subset of X if for each $x \in U$ there exists $\epsilon > 0$ such that

$$B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\} \subset U.$$

Examples $B_r(x) = \{y \in X | d(x, y) < r\}$ is open.

Proposition 1 *Let (X,d) be a metric space. Then*

1. *A union of open subsets is open.*

2. *A finite intersection of open sets is open.*

Proof

1. Let $x \in \bigcup_\alpha U_\alpha$ where U_α is an open subset of X. Then $x \in U_\alpha$ for some α. Thus there exists ϵ such that $B_\epsilon(x) \subset U_\alpha$ and hence $B_\epsilon(x) \subset \bigcup_\alpha U_\alpha$.

2. If suffices to show two sets case. Let U_1, U_2 be open sets in X and $x \in U_1 \cap U_2$. Then there exist ϵ_1, ϵ_2 for such that $B_{\epsilon_1}(x) \subset U_1$, $B_{\epsilon_2}(x) \subset U_2$, respectively. Define $\epsilon = min\{\epsilon_2, \epsilon_2\}$. Then $B_\epsilon(x) \subset U_1 \cap U_2$.

$\square$

Note An infinite intersection of open sets is not necessarily open.

Examples 1. $\cap_{n=1}^{\infty}(-1, \frac{1}{n})$

2. An open set can be viewed simply as a union of open balls.

Continuous functions

Definition 3 Let f be a function form a metric space (X, d_x) to a metric space (Y, d_y). Then

1. f is continuous at $x_0 \in X$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$.

2. f is continuous on X if f is continuous at every $x \in X$

Examples 1. A constant and the identity functions are continuous.

2. The composition of two continuous functions are continuous.

Proposition 2 *Let X and Y be metric spaces. Then $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for all open set V in Y*

Proof ($\Rightarrow$) Let $x \in X$. Since f is continuous, for all ϵ, there exist δ such that $f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset V$. Then $B_\delta(x) \subset f^{-1}(V)$. Thus $f^{-1}(V)$ is open.
($\Leftarrow$) Since $B_\epsilon(f(x))$ is open, $B_\delta(f(x)) \subset f^{-1}(B_\epsilon(f(x)))$ for some δ. Then $f(B_\delta(x)) \subset B_\epsilon(f(x))$. Thus f is continuous. $\square$