I.5 Subspace and Product space

Definition 1 Let X be a topological space and A be a subset in X. Then $\mathcal{T} = \{U \cap A | U \textit{ is open in } X\}$ is called the subspace topology.

Proposition 1 *Let $\mathcal{B}$ be a basis for a space X. Then $\{B \cap A | B \in \mathcal{B}\}$ is a basis for a subspace A.*

Proof First of all, $\cup\{B \cap A\} = A$. Given B_1 and B_2, basis elements for X, and given $a \in A$. Since B_1 and B_2 are bases, there are another basic open set B_3 such that $a \in B_3 \subset B_1 \cap B_2$. Thus $a \in (B_3 \cap A) \subset (B_1 \cap A) \cap (B_2 \cap A)$. Thus $\{B \cap A\}$ satisfies basis axioms. □

Example

1. A basis for $[0, 1] \subset \mathbb{R}$.
2. A basis of $\mathbb{Q}$ is $\mathbb{Q} \cap (a, b)$.

Proposition 2 *Let Y be a subspace of X. If U is open in Y and Y is open in X, then A is open in X.*

Proof Since U is open in Y, $U = Y \cap V$ for some V open in X. Since Y and V are open in X, so is $Y \cap V$. □

Proposition 3 *Let Y be a subspace of X. If C is closed in Y and Y is closed in X, then C is closed in X.*

Proposition 4

1. *Let $f : X \to Y$ be continuous and $A \subset X$, $f(A) \subset Z \subset Y$. Then $f|_A : A \to Z$ is also continuous.*
2. *Let $X = \cup_\alpha X_\alpha$ and X_α is open in X for all α. Then $f : X \to Y$ be continuous if and only if $f|_{X_\alpha} : X_\alpha \to Y$ is also continuous for all α.*

Proof

1. Let V be an open set in Z,i.e, $V = U \cap Z$ for some U open in Y. Then $f|_A^{-1}(V) = f^{-1}(V) \cap A = f^{-1}(U) \cap A$. Since f is continuous, $f^{-1}(U)$ is open in X, and hence $f^{-1}(U) \cap A$ is open in A.
2. ($\Rightarrow$) trivial by 1.
   ($\Leftarrow$) It suffice to show that $f^{-1}(V)$ is open for V open in Y. Observe that $f|_{X_\alpha}^{-1}(V)$ is open in X_α, and hence in X since X_α is open in X. Thus $f^{-1}(V) = \cup_\alpha f|_{X_\alpha}^{-1}(V)$ is open in X.

□

Definition 2 Let A be an index set and X_α be a topological space for $\alpha \in A$. The product of this indexed sets is defined by

$$\prod_{\alpha \in A} X_\alpha = \{x : A \to \coprod_\alpha X_\alpha |\ x(\alpha) \in X_\alpha\}.$$

Example Let $A = \mathbb{N}$ and $X_\alpha = \mathbb{R}$. Then $x : \mathbb{N} \to \mathbb{R}$ is a sequence $x = (x_n)$, $n = 1, 2, ...$

Definition 3 The projection map is defined by

$$p_\alpha : \prod_{\beta \in A} X_\beta \to X_\alpha$$

which assign each element of the product space to its αth coordinate: $x \mapsto x(\alpha)$.

Definition 4 (product topology) Let $X = \prod_\alpha X_\alpha$ and let U_α be an open set in X_α. Then $\mathcal{S} = \{p_\alpha^{-1}(U_\alpha) |\ \alpha \in A \textit{ and } U_\alpha \textit{ open in } X_\alpha\}$ is a subbasis and the topology $\mathcal{T}(\mathcal{S})$ generated by $\mathcal{S}$ is called the product topology of X.

Note $p_\alpha : \prod_{\beta \in A} X_\beta \to X_\alpha$ is continuous with respect to the product topology and it is the coarsest topology with this property.

Proposition 5 *$f : Y \to \prod X_\alpha$ is continuous if and only if $p_\alpha \circ f$ is continuous for each $\alpha \in A$.*

Proof ($\Rightarrow$) The composition of two continuous functions is also continuous. ($\Leftarrow$) It suffices to show that the inverse image of subbasic open set is open. Let U_α be open in X_α. Observe that $f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$,which is open in Y. □

Homework 1 Prove the followings.

1. If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$. Show that the subspace topology of $A \times B$ in $X \times Y$ is the same as the product topology.

2. If $f : X \to Y$ and $g : Z \to W$ are continuous, then $f \times g : X \times Z \to Y \times W$ is continuous.